Applied Finite Mathematics

Third Edition, 2016

Revised and Updated by

Roberta Bloom

De Anza College

Based on Second Edition 1996

By Rupinder Sekhon

De Anza College

SECTION 4.4 PROBLEM SET: CHAPTER REVIEW

Solve the following linear programming problems using the simplex method.

1) Maximize $z = 5x_1 + 3x_2$ subject to $x_1 + x_2 \le 12$ $2x_1 + x_2 \le 16$ $x_1 \ge 0;\ x_2 \ge 0$	2) Maximize $z = 5x_1 + 8x_2$ subject to $x_1 + 2x_2 \le 30$ $3x_1 + x_2 \le 30$ $x_1 \ge 0;\ x_2 \ge 0$
3) Maximize $z = 2x_1 + 3x_2 + x_3$ subject to $4x_1 + 2x_2 + 5x_3 \le 32$ $2x_1 + 4x_2 + 3x_3 \le 28$ $x_1, x_2, x_3 \ge 0$	4) Maximize $z = x_1 + 6x_2 + 8x_3$ subject to $x_1 + 2x_2 \le 1200$ $2x_2 + x_3 \le 1800$ $4x_1 + x_3 \le 3600$ $x_1, x_2, x_3 \ge 0$
5) Maximize $z = 6x_1 + 8x_2 + 5x_3$ subject to $4x_1 + x_2 + x_3 \le 1800$ $2x_1 + 2x_2 + x_3 \le 2000$ $4x_1 + 2x_2 + x_3 \le 3200$ $x_1, x_2, x_3 \ge 0$	6) Minimize $z = 12x_1 + 10x_2$ subject to $x_1 + x_2 \ge 6$ $2x_1 + x_2 \ge 8$ $x_1 \ge 0;\ x_2 \ge 0$
7) Minimize $z = 4x_1 + 6x_2 + 7x_3$ subject to $x_1 + x_2 + 2x_3 \ge 20$ $x_1 + 2x_2 + x_3 \ge 30$ $x_1, x_2, x_3 \ge 0$	8) Minimize $z = 40x_1 + 48x_2 + 30x_3$ subject to $2x_1 + 2x_2 + x_3 \ge 25$ $x_1 + 3x_2 + 2x_3 \ge 30$ $x_1, x_2, x_3 \ge 0$

9) An appliance store sells three different types of ovens: small, medium, and large.
The small, medium, and large ovens require, respectively, 3, 5, and 6 cubic feet of storage space; a maximum of 1,000 cubic feet of storage space is available. Each oven takes 1hour of sales time; there is a maximum of 200 hours of sales labor time available for ovens. The small, medium, and large ovens require, respectively, 1, 1, and 2 hours of installation time; a maximum of 280 hours of installer labor for ovens is available monthly.
If the profit made from sales of small, medium and large ovens is \$50, \$100, and \$150, respectively, how many of each type of oven should be sold to maximize profit, and what is the maximum profit?

5.1 Exponential Growth and Decay Models

In this section, you will learn to

1. *recognize and model exponential growth and decay*
2. *compare linear and exponential growth*
3. *distinguish between exponential and power functions*

COMPARING EXPONENTIAL AND LINEAR GROWTH

Consider two social media sites which are expanding the number of users they have:

- Site A has 10,000 users, and expands by adding 1,500 new users each month
- Site B has 10,000 users, and expands by increasing the number of users by 10% each month.

The number of users for Site A can be modeled as linear growth. The number of users increases by a constant number, 1500, each month. If x = the number of months that have passed and y is the number of users, the number of users after x months is y = 10000+1500x.

For site B, the user base expands by a constant percent each month, rather than by a constant number. Growth that occurs at a constant percent each unit of time is called exponential growth.

We can look at growth for each site to understand the difference. The table compares the number of users for each site for 12 months. The table shows the calculations for the first 4 months only, but uses the same calculation process to complete the rest of the 12 months.

Month	Users at Site A	Users at Site B
0	10000	10000
1	11500 + 1500 = 13000	10000 + 10% of 10000 = 10000 + 0.10(10000) =10000(1.10) = 11000
2	11500 + 1500 = 13000	11000 + 10% of 11000 = 11000 + 0.10(11000) =11000(1.10) = 12100
3	13000 + 1500 = 14500	12100 + 10% of 12100 = 12100 + 0.10(12100) =12100(1.10) = 13310
4	14500 + 1500 = 16000	13310 + 10% of 13310 = 13310 + 0.10(13310) =13310(1.10) = 14641
5	17500	16105
6	19000	17716
7	20500	19487
8	22000	21436
9	23500	23579
10	25000	25937
11	26500	28531
12	28000	31384

PROPERTIES OF EXPONENTIAL DECAY FUNCTIONS

The function $y=f(x) = ab^x$ function represents decay if $0 < b < 1$ and $a > 0$.

The growth rate r is negative when $0 < b<1$. Because $b = 1+ r < 1$, then $r = b–1 < 0$.

The function $y=f(x) = ae^{kx}$ function represents decay if $k < 0$ and $a > 0$.

The function is a decreasing function; y decreases as x increases.

Domain: { all real numbers} ; all real numbers can be input to an exponential function

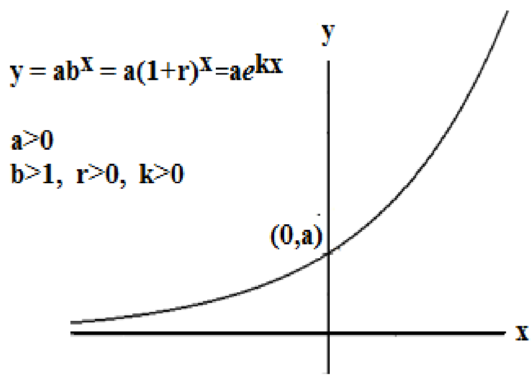
Range: If $a>0$, the range is {positive real numbers}
The graph is always above the x axis.

Horizontal Asymptote: when $b < 1$, the horizontal asymptote is the positive x axis as x becomes large positive. Using mathematical notation: as $x \to \infty$, then $y \to 0$

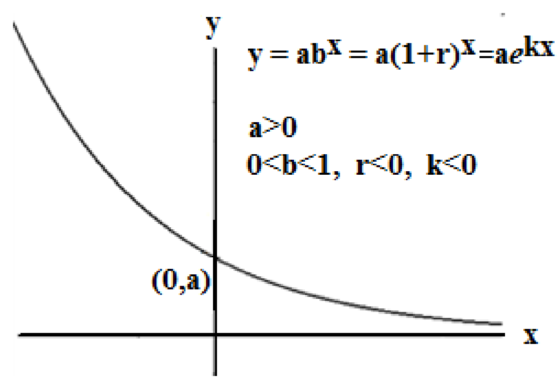
The vertical intercept is the point (0,a) on the y-axis.
There is no horizontal intercept because the function does not cross the x-axis.

The graphs for exponential growth and decay functions are displayed below for comparison

EXPONENTIAL GROWTH



EXPONENTIAL DECAY



AN EXPONENTIAL FUNCTION IS A ONE-TO-ONE FUNCTION

Observe that in the graph of an exponential function, each y value on the graph occurs only once. Therefore, every y value in the range corresponds to only one x value. So, for any particular value of y, you can use the graph to see which value of x is the input to produce that y value as output. This property is called "**one-to-one**".

Because for each value of the output y, you can uniquely determine the value of the corresponding input x, thus every exponential function has an inverse function.
The inverse function of an exponential function is a logarithmic function, which we will investigate in the next section.

◆***Example 8*** a. Express $y = 4200\,(1.078)^t$ in the form $y = ae^{kt}$
b. Express $y = 150\,(0.73)^t$ in the form $y = ae^{kt}$

Solution: a. Express $y = 4200\,(1.078)^t$ in the form $y = ae^{kt}$

$$y = ae^{kt} = ab^t$$
$$a(e^k)^t = ab^t$$
$$e^k = b$$
$$e^k = 1.078$$

Therefore $k = \ln 1.078 \approx 0.0751$

We rewrite the growth function as $y = 3500e^{0.0751t}$

b. Express $y = 150\,(0.73)^t$ in the form $y = ae^{kt}$

$$y = ae^{kt} = ab^t$$
$$a(e^k)^t = ab^t$$
$$e^k = 0.73$$
$$e^k = 0.73$$

Therefore $k = \ln 0.73 \approx -0.3147$

We rewrite the growth function as $y = 150\,e^{-0.3147t}$

AN APPLICATION OF A LOGARITHMIC FUNCTON

Suppose we invest $10,000 today and want to know how long it will take to accumulate to a specified amount, such as $15,000. The time t needed to reach a future value y is a logarithmic function of the future value: t = g(y)

◆***Example 9*** Suppose that Vinh invests $10000 in an investment earning 5% per year. He wants to know how long it would take his investment to accumulate to $12000, and how long it would take to accumulate to $15000.

Solution: We start by writing the exponential growth function that models the value of this investment as a function of the time since the $10000 is initially invested

$$y = 10000(1.05)^t$$

We divide both sides by 10000 to isolate the exponential expression on one side.

$$\frac{y}{10000} = 1.05^t$$

Next we rewrite this in logarithmic form to express time as a function of the accumulated future value. We'll use function notation and call this function g(y).

$$t = g(y) = \log_{1.05}\left(\frac{y}{10000}\right)$$

◆***Example 1*** Ursula borrows \$600 for 5 months at a simple interest rate of 15% per year. Find the interest, and the total amount she is obligated to pay?

Solution: The interest is computed by multiplying the principal with the interest rate and the time.

$$I = Prt$$
$$I = \$600(.15)\frac{5}{12} = \$37.50$$

The total amount is

$$A = P + I = \$600 + \$37.50 = \$637.50$$

Incidentally, the total amount can be computed directly as

$$\begin{aligned} A = P(1 + rt) &= \$600[1 + (.15)(5/12)] \\ &= \$600(1 + .0625) \\ &= \$637.50 \end{aligned}$$

◆***Example 2*** Jose deposited \$2500 in an account that pays 6% simple interest. How much money will he have at the end of 3 years?

Solution: The total amount or the future value is given by $A = P(1 + rt)$.

$$A = \$2500[1 + (.06)(3)]$$
$$A = \$2950$$

◆***Example 3*** Darnel owes a total of \$3060 which includes 12% interest for the three years he borrowed the money. How much did he originally borrow?

Solution: This time we are asked to compute the principal P.

$$\$3060 = P[1 + (.12)(3)]$$
$$\$3060 = P(1.36)$$
$$\frac{\$3060}{1.36} = P$$
$$\$2250 = P$$

Darnel originally borrowed \$2250.

◆***Example 4*** A Visa credit card company charges a 1.5% finance charge each month on the unpaid balance. If Martha owed \$2350 and has not paid her bill for three months, how much does she owe now?

Solution: Before we attempt the problem, the reader should note that in this problem the rate of finance charge is given per month and not per year.

The total amount Martha owes is the previous unpaid balance plus the finance charge.

$$A = \$2350 + \$2350(.015)(3) = \$2350 + \$105.75 = \$2455.75$$

Alternatively, again, we can compute the amount directly by using formula $A = P(1 + rt)$

$$A = \$2350[1 + (.015)(3)] = \$2350(1.045) = \$2455.75$$

◆***Example 5*** A state issues a 15 year \$1000 bond that pays \$25 every six months. If the current market interest rate is 4%, what is the fair market value of the bond?

Solution: The bond certificate promises two things – an amount of \$1,000 to be paid in 15 years, and semi-annual payments of \$25 for 15 years. To find the fair market value of the bond, we find the present value of the \$1,000 face value we are to receive in 15 years and add it to the present value of the \$25 semi-annual payments for the 15 years. In this example, nt = 2(15)=30.

We will let P_1 = the present value of the lump-sum \$1,000

$$P_1(1 + .04/2)^{30} = \$1,000$$

$$P_1 = \$552.07$$

We will let P_2 = the present value of the \$25 semi-annual payments is

$$P_2\,(1 + .04/2)^{30} = \frac{\$25[(1+.04/2)^{30} - 1]}{(.04/2)}$$

$$P_2\ (1.18114) = \$1014.20$$

$$P_2 = \$559.90$$

The present value of the lump-sum \$1,000 = \$552.07

The present value of the \$30 semi-annual payments = \$559.90

Therefore, the fair market value of the bond is

$$P = P_1 + P_2 = \$552.07 + \$559.90 = \$1111.97$$

Because the market interest rate of 4% is lower than the interest rate of 5% implied by the semiannual payments, the bond is selling at a premium: the fair market value of \$1,111.97 is more than the face value of \$1,000.

To summarize:

To Find the Fair Market Value of a Bond:

Find the present value of the face amount A that is payable at the maturity date:

$\mathbf{A = P_1\,(1 + r/n)^{nt}}$; solve to find P_1

Find the present value of the semiannually payments of \$m over the term of the bond:

$\mathbf{P_2(1 + r/n)^{nt} = \dfrac{m[(1 + r/n)^{nt} - 1]}{r/n}}$; solve to find P_2

The fair market value (or present value or price or current value) of the bond is the sum of the present values calculated above:

$P = \mathbf{P_1 + P_2}$

SECTION 7.3 PROBLEM SET: PERMUTATIONS

Do the following problems using permutations.

17) A bakery has 9 different fancy cakes. In how many ways can 5 of the 9 fancy cakes be lined up in a row in the bakery display case?	18) A landscaper has 6 different flowering plants. She needs to plant 4 of them in a row in a garden. How many different ways can 4 of the 6 plants be arranged in a row?
19) At an auction of used construction vehicles, there are 7 different vehicles for sale. In how many orders could these 7 vehicles be listed in the auction program?	20) A landscaper has 6 different flowering plants and 4 different non-flowering bushes. She needs to plant a row of 6 plants in a garden. There must be a bush at each end, and four flowering plants in a row in between the bushes. How many different arrangements in a row are possible?
21) In how many ways can all 7 letters of the word QUIETLY be arranged if the letters Q and U must be next to each other in the order QU?	22) a. In how many ways can the letters ABCDEXY be arranged if the X and Y must be next to each other in either order XY or YX? b. In how many ways can the letters ABCDEXY be arranged if the X and Y can not be next to each other?

SECTION 7.8 PROBLEM SET: CHAPTER REVIEW

16) In how many ways can 3 books be selected from 4 English and 2 History books if at least one English book must be chosen?

17) Five points lie on the rim of a circle. Choosing the points as vertices, how many different triangles can be drawn?

18) A club consists of six men and nine women. In how many ways can a president, a vice president and a treasurer be chosen if the two of the officers must be women?

19) Of its 12 sales people, a company wants to assign 4 to its Western territory, 5 to its Northern territory, and 3 to its Southern territory. How many ways can this be done?

20) How many permutations of the letters of the word OUTSIDE have consonants in the first and last place?

21) How many distinguishable permutations are there in the word COMMUNICATION?

22) How many five-card poker hands consisting of the following distribution are there?
    a) A flush(all five cards of a single suit)
    b) Three of a kind(e.g. three aces and two other cards)
    c) Two pairs(e.g. two aces, two kings and one other card)
    d) A straight(all five cards in a sequence)

23) Company stocks on an exchange are given symbols consisting of three letters. How many different three-letter symbols are possible?

24) How many four-digit odd numbers are there?

25) In how many ways can 7 people be made to stand in a straight line? In a circle?

26) A United Nations delegation consists of 6 Americans, 5 Russians, and 4 Chinese. Answer the following questions.
    a) How many committees of five people are there?
    b) How many committees of three people consisting of at least one American are there?
    c) How many committees of four people having no Russians are there?
    d) How many committees of three people have more Americans than Russians?
    e) How many committees of three people do not have all three Americans?

27) If a coin is flipped five times, in how many different ways can it show up three heads?

28) To reach his destination, a man is to walk three blocks north and four blocks west. How many different routes are possible?

29) All three players of the women's beach volleyball team, and all three players of the men's beach volleyball team are to line up for a picture with all members of the women's team lined together and all members of men's team lined up together. How many ways can this be done?

30) From a group of 6 Americans, 5 Japanese and 4 German delegates, two Americans, two Japanese and a German are chosen to line up for a photograph. In how many different ways can this be done?

31) Find the fourth term of the expansion $(2x - 3y)^8$.

32) Find the coefficient of the a^5b^4 term in the expansion of $(a – 2b)^9$.

SECTION 8.5 PROBLEM SET: INDEPENDENT EVENTS

The distribution of the number of fiction and non-fiction books checked out at a city's main library and at a smaller branch on a given day is as follows.

	MAIN (M)	BRANCH (B)	TOTAL
FICTION (F)	300	100	400
NON-FICTION (N)	150	50	200
TOTALS	450	150	600

Use this table to determine the following probabilities:

1) P(F)	2) P(M | F)
3) P(N | B)	4) Is the fact that a person checks out a fiction book independent of the main library? Use probabilities to justify your conclusion.

For a two-child family, let the events E, F, and G be as follows.

E: The family has at least one boy
F: The family has children of both sexes
G: The family's first born is a boy

5) Find the following. a) P(E) b) P(F) c) $P(E \cap F)$ d) Are E and F independent? Use probabilities to justify your conclusion.	6) Find the following. a) P(F) b) P(G) c) $P(F \cap G)$ d) Are F and G independent? Use probabilities to justify your conclusion.

SECTION 8.5 PROBLEM SET: INDEPENDENT EVENTS

13) John's probability of passing statistics is 40%, and Linda's probability of passing the same course is 70%. If the two events are independent, find the following probabilities.
    a) P(both of them will pass statistics)
    b) P(at least one of them will pass statistics)

14) Jane is flying home for the Christmas holidays. She has to change planes twice. There is an 80% chance that she will make the first connection, and a 90% chance that she will make the second connection. If the two events are independent, find the probabilities:
    a) P(Jane will make both connections)
    b) P(Jane will make at least one connection)

For a three-child family, let the events E, F, and G be as follows.

E: The family has at least one boy
F: The family has children of both sexes
G: The family's first born is a boy

15) Find the following.
    a) P(E)
    b) P(F)
    c) $P(E \cap F)$
    d) Are E and F independent?

16) Find the following.
    a) P(F)
    b) P(G)
    c) $P(F \cap G)$
    d) Are F and G independent?

SECTION 9.3 PROBLEM SET: EXPECTED VALUE

13) In a housing development, 35% of households have no school age children, 20% of households have 1 school age child, 25% of households have 2 school age children, 15% have 3, and 5% have 4 school age children.

a. Find the average number of children per household

b. If there are 300 homes in this housing development, what is the total number of children expected to attend school?

14) At a large community college, 30% of students take one course, 15% take two courses, 25% take three courses and 20% take four courses. The rest of the students take five courses.

a. What percent of students take 5 courses?

b. Find the average number of courses that students take.

.

SECTION 10.2 PROBLEM SET: APPLICATIONS OF MARKOV CHAINS

Questions 1-2 refer to the following:

Reference: Bart Sinclair, Machine Repair Model. OpenStax CNX. Jun 9, 2005 Creative Commons Attribution License 1.0. Download for free at http://cnx.org/contents/56f1bed0-bd34-4c28-a2ec-4a3f9ded8e18@3. This material has been modified by Roberta Bloom, as permitted under that license.

A Markov chain can be used to model the status of equipment, such as a machine used in a manufacturing process. Suppose that the possible states for the machine are

Idle & awaiting work (I)   Working on a job/task (W)   Broken (B)   In Repair (R)

The machine is monitored at regular intervals to determine its status; for ease of interpretation in this problem, we assume the status is monitored every hour. The transition matrix is shown below.

$$T = \begin{array}{c} \\ I \\ W \\ B \\ R \end{array} \begin{array}{c} \begin{array}{cccc} I & W & B & R \end{array} \\ \begin{bmatrix} 0.05 & 0.93 & 0.02 & 0 \\ 0.10 & 0.86 & 0.04 & 0 \\ 0 & 0 & 0.80 & 0.20 \\ 0.5 & 0.1 & 0 & 0.4 \end{bmatrix} \end{array}$$

1. Use the transition matrix to identify the following probabilities concerning the state of the machine one hour from now	2. Perform the appropriate calculations using the transition matrix to find the following probabilities concerning the state of the machine three hours from now.
a) Find the probability that the machine is working on a job one hour from now if the machine is idle now.	a) Find the probability that the machine is working on a job three hours from now if the machine is idle now.
b) Find the probability that the machine is idle one hour from now if the machine is working on a job now.	b) Find the probability that the machine is idle three hours from now if the machine is working on a job now.
c) Find the probability that the machine is working on a job one hour from now if the machine is being repaired now.	c) Find the probability that the machine is working on a job three hours from now if the machine is being repaired now.
d) Find the probability that the machine is being repaired in one hour if it is broken now.	d) Find the probability that the machine is being repaired in three hours if it is broken now.

SECTION 10.2 PROBLEM SET: APPLICATIONS OF MARKOV CHAINS

Question 5 refers to the following:

Markov chains play an important role in online search.

"PageRank is an algorithm used by Google Search to rank websites in their search engine results. PageRank was named after Larry Page, one of the founders of Google. PageRank is a way of measuring the importance of website pages"
Source: https://en.wikipedia.org/wiki/PageRank under the Creative Commons Attribution-ShareAlike License;

The theory behind PageRank is that pages that are linked to more often are more important and useful; identifying those that are linked to more often about a topic helps identify the pages that should be presented as most pertinent in a search.

In real world search, there are thousands or millions of pages linking together, resulting in huge transition matrices. Because of the size and other properties of these matrices, the mathematics behind PageRank is more sophisticated than the small example we examine here with only four websites. However our example is adequate to convey the main concept of PageRank and its use in search algorithms.

It should be noted that real world search algorithms, PageRank or similar Markov chain ranking schemes are only one of a variety of methods used.

Suppose we have 4 webpages that contains links to each other. We call the pages A, B, C, D.

- From page A, 30% of people link to page B, 50% link to page C, and 20% link to page D
- From page B, 50% of popele link to page A and 50% link to page D
- From page C, 10% of people link to page B, 70% link to page C, and 20% link to page D
- From page D, 20% of people link to page A, 40% to page B, 10% to page C, and 30% link to page D

(In this example, when a page links to itself, it means that a person viewing the page stays at that page and does not link to another page.)

a) Write the transition matrix T	b) Find the probability that a person viewing page C will link to page D next.
c) Find the probability that a person viewing page C will view page D after two links	d) Find the probability that a person viewing page C will view page D after **three** links
e) Find the probability that a person viewing page C will stay at page C and not link to any other page next.	f) Find the probability that a person viewing page C will view page A next

◆***Example 6*** At a professional school, students need to take and pass an English writing/speech class in order to get their professional degree. Students must take the class during the first quarter that they enroll. If they do not pass the class they take it again in the second semester. If they fail twice, they are not permitted to retake it again, and so would be unable to earn their degree.

Students can be in one of 4 states: passed the class (P), enrolled in the class for the first time (C), retaking the class (R) or failed twice and can not retake (F). Experience shows 70% of students taking the class for the first time pass and 80% of students taking the class for the second time pass.

Write the transition matrix and identify the absorbing states.
Find the probability of being absorbed eventually in each of the absorbing states.

Solution: The absorbing states are P (pass) and F (fail repeatedly and can not retake).
The transition matrix T is shown below.

$$T = \begin{array}{c} \\ P \\ C \\ R \\ F \end{array} \begin{array}{c} \begin{array}{cccc} P & C & R & F \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ .7 & 0 & .3 & 0 \\ .8 & 0 & 0 & .2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

If we raise the transition matrix T to a high power, we find that it remains stable and gives us the long-term probabilities of ending up in each of the absorbing states.

$$T^{30} = \begin{array}{c} \\ P \\ C \\ R \\ F \end{array} \begin{array}{c} \begin{array}{cccc} P & C & R & F \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ .94 & 0 & 0 & .06 \\ .8 & 0 & 0 & .2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Of students currently taking the class for the first time, 94% will eventually pass. 6% will eventually fail twice and be unable to earn their degree.

Of students currently taking the class for the second time time, 80% will eventually pass. 20% will eventually fail twice and be unable to earn their degree.

The solution matrix contains the same information in abbreviated form

$$\text{Solution Matrix} = \begin{array}{c} \\ C \\ R \end{array} \begin{array}{c} \begin{array}{cc} P & F \end{array} \\ \begin{bmatrix} .94 & .06 \\ .80 & .20 \end{bmatrix} \end{array}$$

Note that in this particular problem, we don't need to raise T to a "very high" power. If we find T^2, we see that it is actually equal to T^n for higher powers n. T^n becomes stable after two transitions; this makes sense in this problem because after taking the class twice, the student must have passed or is not permitted to retake it any longer. Therefore the probabilities should not change any more after two transitions; by the end of two transitions, every student has reached an absorbing state.